Multicritical phases of the $O(n)$ model on a random lattice

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Multicritical Phases of the O(n) Model on a Random Lattice

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We exhibit the multicritical phase structure of the loop gas model on a random surface. The dense phase is reconsidered, with special attention paid to the topological points $g = 1/p$. This phase is complementary to the dilute and higher multicritical phases in the sense that dense models contain the same spectrum of bulk operators (found in the continuum by Lian and Zuckerman) but a different set of boundary operators. This difference illuminates the well-known (p, q) asymmetry of the matrix chain models. Higher multicritical phases are constructed, generalizing both Kazakov's multicritical models as well as the known dilute phase models. They are quite likely related to multicritical polymer theories recently considered independently by Saleur and Zamolodchikov. Our results may be of help in defining such models on *flat* honeycomb lattices; an unsolved problem in polymer theory. The phase boundaries correspond again to "topological" points with $g = p/1$ integer, which we study in some detail. Two qualitatively different types of critical points are discovered for each such g. For the special point $g = 2$ we demonstrate that the dilute phase $O(-2)$ model does *not* correspond to the Parisi-Sourlas model, a result likely to hold as well for

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the flat case. Instead it is proven that the first *multicritical* $O(-2)$ point possesses the Parisi-Sourlas supersymmetry.

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1. Introduction and Overview

1.1. Introduction

The past few years have seen considerable progress towards understanding theories of conformal matter coupled to 2D gravity. Although some insights were due to advances in continuum Liouville theory, stunning progress was made in understanding and solving *discrete* models of 2D gravity. Initially, the number of such models available was small. Due to recent interest in the subject, however, we now have a plethora of such models available. Some of these models are known to correspond to conformally coupled matter of central charges $C \leq 1$, while the continuum limit of many others still remains to be understood. Classifying the possible critical behaviour of lattice models of 2D gravity is potentially an important pursuit. Because of their interpretation as toy models of bosonic string theory we might hope to eventually attack more physical string theories with higher central charges and built-in supersymmetry.

The perhaps simplest such theory is the one matrix model and its multicritical points discovered by Kazakov [1]. They were identified in [2], [3]as corresponding to the $(2m-1,2)$ minimal models coupled to gravity. Even though this identification is beyond any doubt correct, it is less clear why the lattice curvature dependent Boltzmann weights of these models conspire to give precisely these minimal models. In fact, we only understand this puzzle for the first three cases $m = 1,2$ and $m = 3$ [2]. General (p,q) minimal models require several linearly coupled matrices [4]; actually two are sufficient [5]. Again a better understanding of why the Boltzmann weights of these multimatrix models result in particular conformally coupled theories would be quite interesting.

An alternative way to introduce $C \leq 1$ matter fields onto random lattices is the loop gas construction [6][7]. An advantage over the multimatrix approach is that the construction is manifestly physical; i.e. it is clear (through the Coulomb gas mapping) why a specific critical behavior occurs. All (p,q) minimal models coupled to quantum gravity can be constructed in this way. So far a non-perturbative definition of all these models has not been given; it is however possible to derive diagrammatic rules signaling the existence of a "string field theory" [8].

The simplest statistical model allowing interpretation in terms of the loop gas is the $O(n)$ model [9]. The $O(n)$ on a fluctuating surface is equivalent to a special one-matrix model [6] and thus can be defined in a nonperturbative way. In this simplest model all loops (contractible and noncontractible) are taken with the same weight n. The partition function of the corresponding loop gas on a 3-coordinated fluctuating lattice reads

$$F(\lambda, T, N) = \sum_{\phi^3 \text{graphs}} \sum_{\text{loops}} N^{2-2H} n^{\#\text{loops}} e^{-\lambda v} e^{-T(\text{total length of loops})} \tag{1.1}$$

where λ is the cosmological constant coupled to the volume $v = \#$ vertices of the lattice, T is the temperature of the loop gas and $1/N$ is the string interaction constant coupled to the Euler characteristics $2 - 2H$ of the graph. The right hand side of (1.1) represents a triple series: It is asymptotic in $1/N^2$ and of finite radius of convergence in the fugacities $g_1 = e^{-\lambda}$ and $\frac{1}{2P_0} = e^{-\lambda - T}$ corresponding to vertices empty and occupied by loops, respectively.

For N and n integers this series coincides with the perturbative expansion of the vacuum energy of a zero-dimensional $N \times N$ matrix field theory [6]

$$Z = \int \mathcal{D}M \prod_{i=1}^{n} \mathcal{D}\Phi_i \exp\{-N\text{Tr}[V(M) + \frac{1}{2}\sum_{\mu=1}^{n} \Phi_\mu^2 - \frac{1}{2P_0}\sum_{\mu=1}^{n} \Phi_\mu^2 M]\} \tag{1.2}$$

The corresponding Feynman diagrams can be interpreted in terms of surfaces populated by nonintersecting loops (fig. 1) Then n gives the number of species of loops and the matrix potential $V(M)$ specifies the measure in the configuration space of the random surface. With the choice $V(M) = \frac{1}{2}M^2 + \frac{1}{3}g_1 M^3$ the perturbative expansion of the matrix integral creates the cluster expansion of the $O(n)$ model on a 3-coordinated lattice.

The integration over the matrix variables $\Phi_\mu, \mu = 1, 2, ..., n$ can be performed immediatelly and the result is the following one-matrix integral [6]

$$Z = \int \mathcal{D}M \exp\{-N\text{Tr } V(M) + \frac{n}{2}\int_0^\infty \frac{d\ell}{\ell}(\text{Tr } e^{M\ell})^2\} \tag{1.3}$$

The method of orthogonal polynomials used in the "ordinary" matrix models is not applicable here but the saddle point method in the limit $N \to \infty$ works equally well. Indeed, the action depends only on the eigenvalues $P_1, ..., P_N$ of the random matrix M and the integral (1.3) equals, up to a constant factor, to

$$Z = \int \prod_{i=1}^{N} dP_i \; e^{-N\sum_{i=1}^{N} V(P_i) \; + \sum_{i\neq j} \log|P_i - P_j| - \frac{n}{2}\sum_{i,j} \log|2P_0 - P_i - P_j|} \tag{1.4}$$

This last integral is defined for any real value of n and it is known from the flat case that criticality occurs for $-2 \leq n \leq 2$. This is exactly the interval in which the saddle point solution exists [6]. In this paper we will restrict our study to the saddle point solution; this is sufficient for classifying the possible critical regimes of the model. The stability if the saddle point solution and the nonperturbative effects due to instantons are considered in [10].

It is very useful to parametrize n in terms of the Coulomb gas coupling constant g as

$$n = -2\cos\pi g \tag{1.5}$$

The model can be in a dense phase with $0 \leq g \leq 1$ or, if g_1 in $V(M)$ is tuned appropriately, in a dilute phase with $1 \leq g \leq 2$. In both phases the central charge is given by $C = 1 - 6(\sqrt{g} - \frac{1}{\sqrt{g}})^2$. Which models may be described within the two phases? Let us agree that for our entire discussion in the present work $p > q$. Then in the dense phase $g = \frac{q}{p}$ and all minimal[1] models may be obtained; while in the dilute phase $g = \frac{p}{q}$ and only models with $p < 2q$ can be reached. In particular, the $p = 2m-1$,$q = 2$ models are not present for $m \geq 3$. It is therefore natural to put them in by hand and investigate the loop gas in the presence of a general potential

$$V(M) = \sum_{k=2}^{m+1} \frac{1}{k} g_k M^k \tag{1.6}$$

In order to analyze the possible critical behavior it is sufficient to study the theory on a disc, i.e. we will consider the loop function

$$W(P) = \langle \frac{1}{N} \mathrm{Tr} \frac{1}{\mathrm{P} - \mathrm{M}} \rangle \tag{1.7}$$

where the average is taken with respect to (1.2). For later use we remind the reader that the asymptotics of $W(P)$ for $P \to \infty$ is given by

$$W(P) = \frac{1}{P} + \frac{W_1}{P} + \frac{W_2}{P} + \ldots \tag{1.8}$$

1.2. Overview, Conclusions and Open Problems

Our main result consists in establishing the phase diagram of fig. 2. In all phases the relation $n = -2\cos\pi g$ remains valid; i.e. the phases may be thought of as an infinite

[1] Note that the $O(n)$ model is not minimal at generic values of n, even if the associated central charge is in the BPZ list. Special weights have to be assigned to noncontractible loops, as soon as the genus exceeds zero or operator insertions are present (see [7]). In particular eq.(1.2)may not be used to give a nonperturbative definition for the minimal models.

number of branches in the complex n-plane (the branchpoints being at $n = \pm 2$). The Kazakov multicritical points are situated by construction at the center (i.e. $n = 0$) of each phase. For a given central charge between $-\infty$ and 1 there are always two points; one lying in the dense and the other in a higher phase. These two models related by the duality $g \to \frac{1}{g}$ are however *not* equivalent. This is particularly surprising for the minimal models: There is a dense realization at $g = \frac{q}{p}$ and a multicritical one at $g = \frac{p}{q}$. We argue in section 2.2. that the difference is due to a different set of *boundary* operators. From this point of view, then, the infamous (p, q) asymmetry of the matrix chain models arises because the chain models always favor one set of boundary operators over the other (with the exception of the Hermitian one matrix model with $g = \frac{1}{2}$). Another sequence of particularly interesting points in the dense phase is generated by the "topological" series $g = \frac{1}{p}$. We demonstrate that these models appear in our lattice approach as theories of "boundaries" (i.e. loops) without "worldsheet" [2]. Our understanding is aided by the realization that the loop equation of the model may actually be *exactly* solved for rational values of g. The first member ($g = \frac{1}{2}$) of this series is the model of topological gravity whose continuum limit is known [12]. The correct continuum limit of the higher members of the series remains to be found. In section 3. we further establish the above diagram by investigating the "perturbation" of the $(2m-1, 2)$ models by the loop gas. Note that our approach smoothly interpolates between the physical construction in the dense and dilute phase and the higher Kazakov points. One may therefore hope to eventually obtain a better understanding of the latter. Let us note that flat multicritical $O(n)$ models have recently attracted some attention in relation to a search for multicritical theories of polymers [13], [14]. Polymers are obtained by taking a $n \to 0$ limit in the $O(n)$ model. Dense and dilute polymers have been related to a, respectively, broken and unbroken twisted $N = 2$ supersymmetry [13], and it is natural to ask whether some insights into 2D gravity and noncritical string theory can be derived from this fact. A further very interesting problem is to find out what kind of multicritical polymer theories are obtained by taking an $n \to 0$ limit in the Kazakov backgrounds. A comparison of critical exponents (see 3.3.) indicates that our multicritical polymers do not correspond to the ones considered in [13]. In section 4. the phase boundaries between the various regimes are studied. A discontinuity at these points is detected, resulting in two qualitatively different models at each integer g. The first boundary, $g = 1$, possesses $C = 1$, while the second boundary, $g = 2$, has $C = -2$.

[2] This way of looking at topological theories was first clearly stated in [8],[11].

We demonstrate that the model connected to the $m = 3$ phase possesses the Parisi-Sourlas "target space" supersymmetry. The dilute $O(-2)$ model is different; only a $n \to -2$ *limit* in the dilute phase yields again the Parisi-Sourlas model. The further boundaries at $g = 3, 4, \ldots$ all correspond to "topological" models. Their correct continuum description remains obscure for the moment.

2. The Dense Phase, Revisited

2.1. The Loop Equation and its Solution

The dense phase corresponds to setting all couplings $\{g_k\}$ in (1.6) to zero, except for $g_0 = 1$. All loops are then densely packed on the φ^3 random graphs. The loop equation reads

$$W^2(P) = \oint_{\mathcal{C}} \frac{dP'W(P')}{2\pi i(P-P')}[V'(P') - nW(2P_0 - P')] \tag{2.1}$$

It may be derived from the matrix model (1.2)or by combinatorial methods [7]. After a symmetrization with respect to the reflection $P \to 2P_0 - P$ the contour integral can be performed using the Cauchy theorem and we arrive at the following functional equation (for the gaussian potential $V(P) = \frac{1}{2}P^2$)

$$W(P)^2 + W(2P_0 - P)^2 + nW(P)W(2P_0 - P) = PW(P) + (2P_0 - P)W(2P_0 - P) - 2 \tag{2.2}$$

The critical point is located to be at $P_* = \sqrt{2(2+n)}$. If we transform $P = P_0 + \bar{P}$ and $W(P) = \frac{1}{2\sin^2 \pi g}[P + \cos \pi g \ \ (2P_0 - P)] + \bar{W}(\bar{P})$ (2.2)simplifies to

$$\bar{W}(\bar{P})^2 + \bar{W}(-\bar{P})^2 + n\bar{W}(\bar{P})\bar{W}(-\bar{P}) = \frac{1}{2+n}(P_0^2 - P_*^2) + \frac{1}{2-n}\bar{P}^2 \tag{2.3}$$

In [15],[16] the equation was solved in the vicinity of P_*. After introducing a cutoff $\bar{a}$ we blow up the vicinity of P_* by the following change of variables

$$\frac{P_0 - P_*}{P_*} = \bar{a}^{2g}\Lambda, \quad \frac{P - P_0}{P_* - P_L} = \bar{a}z, \quad \frac{P_0 - P_R}{P_* - P_L} = \bar{a}M \tag{2.4}$$

Here Λ is the continuum cosmological constant, z the continuum boundary cosmological constant, and M is the boundary cosmological constant induced by the fluctuations of the worldsheet geometry.

One obtains for the singular part $\bar{a}^g w(z)$ of $W(P)$

$$
\begin{aligned}
w(z) &= -\frac{1}{2}A_g[(z+\sqrt{z^2-M^2})^g + (z-\sqrt{z^2-M^2})^g] \\
&= A_g\ M^g\ \cosh(g\tau); \\
z &= M\ \cosh(\tau)
\end{aligned}
\tag{2.5}
$$

with

$$\Lambda = \frac{2}{(1-g)^2}M^{2g} \tag{2.6}$$

$$A_g = \frac{-2\sqrt{2}}{1-g}\ \ \frac{1}{\sin \pi g} \tag{2.7}$$

The last relation follows if we compare (2.5) with the exact solution obtained for $P_0 = P_*$ in [17]

$$
\begin{aligned}
\bar{W} &= -\frac{1}{2}A_g\ \frac{u^{1-g}+u^{g-1}}{u+u^{-1}} \\
\bar{P} &= \frac{2(P_*-P_L)}{u+u^{-1}} \\
P_* - P_L &= \frac{\sqrt{2(2-\beta)}}{1-g}
\end{aligned}
\tag{2.8}
$$

The inverse Laplace image of $w(z)$ gives the amplitude $\tilde{w}(\ell)$ for a disc with fixed length ℓ of its boundary

$$w(z) = \int_0^\infty d\ell e^{-z\ell}\tilde{w}(\ell);$$

$$\tilde{w}(l) = \frac{2^{3/2}gM^g}{(1-g)}\frac{1}{\ell}K_g(\ell) \sim \begin{cases} \frac{2g\sqrt{\pi}}{(1-g)}M^{g-1/2}\ell^{-3/2}e^{-M\ell}, & \text{if } \ell \gg 1/M, \\ -\frac{2^{g+1/2}\pi g}{(1-g)\sin \pi g}M^{-g}\ell^{-g-1}, & \text{if } \ell \ll 1/M, \end{cases} \tag{2.9}$$

Now (2.5), (2.6) is quite puzzling for several reasons:

1. (2.6) says that quantum area *does not* scale like the square of quantum length in these models. An attempt at an explanation is made in section 2.2.

2. For the topological models $g = \frac{1}{p}$ we see that the string susceptibility is an integer: $\gamma_s = 1 - \frac{1}{g}$. This is surprising, since it means *either* that the exact lattice solution is analytic in the lattice cosmological constant P_0 around P_* *or* that we have logarithmic scaling violations at these points. Below we will argue that the first possibility holds (in accordance with [18]).

3. A subtle but important point is that, for $g < \frac{1}{2}$, we have to approach the critical value of the boundary cosmological constant as $P = P_0 + \bar{a}z$, not $P = P_* + \bar{a}z$. We do

not have a deep explanation for this, but note that it constitutes annother instance of an "analytical redefinition" (discovered in [11]) necessary to get sensible results from lattice gravity models.

It is interesting to realize that (2.2)may actually be exactly solved for *rational* g. This serves as a check for (2.5), (2.6)and will help in getting some insight into the second problem just mentioned. At the critical point $P_0 = P_*$ the solution to (2.3)has been known already for some time [17][7]. It may be parametrized using circular functions or rational functions($g = \frac{q}{p}$):

$$\begin{aligned} \bar{W} &= -\frac{\sqrt{2}}{1-g} \frac{1}{\sin \pi g} \frac{t^{2p-q} + t^q}{t^{2p}+1} \\ \bar{P} &= \frac{4}{1-g} \sqrt{1 + \cos \pi g} \frac{t^p}{t^{2p}+1} \end{aligned} \tag{2.10}$$

Here t is defined on (the double cover of) the Riemann sphere. It is apparent from (2.10)that the Riemann surface of $W(P)$ is *algebraic*. Furthermore, since the surface is uniformized by rational functions of t we see that it has genus zero. P is defined on a surface with p sheets. On the lowest (physical) sheet there is one cut from P_L (branchpoint of order 2) to $P_R = P_0$ (branchpoint of order p). The higher (unphysical) sheets possess also a cut from P_0 to $2P_0 - P_L$, except for the topmost, which again has just one cut. Now it is important to understand what happens if we move away from the critical point: The degeneracy at $P = P_0$ will be removed and all branchpoints will be of order 2. However, we still have a Riemann surface; it has a finite number of sheets and only square root singularities. By a standard theorem we therefore conclude that the solution of (2.3)is algebraic and thus given by

$$\bar{W}^p + h_{p-1}(\bar{P})\bar{W}^{p-1} + \ldots + h_1(\bar{P})\bar{W} + h_0(\bar{P}) = 0 \tag{2.11}$$

Here the $h_i(\bar{P})$ are *polynomials* in $\bar{P}$, with P_0 dependent coefficients. These coefficients are determined by expanding (2.3), (2.11)around $\bar{P} = \infty$ and matching the two expansions. It is easy to see that the genus of the Riemann surface will not change off the critical point; we therefore know the existence of a parametrization of (2.11)using rational functions of one parameter. Unfortunately we did not succeed in finding a simple such parametrization for arbirary g. Nevertheless we are now in a position to get some insight into the second problem mentioned above. Given the just presented algorithm for generating exact solutions to the loop equation we see that the singularities in the lattice cosmological constant P_0 are necessarily *algebraic* for rational g. Logarithmic scaling violations are thus impossible and we have *analytic* behavior in P_0 at the topological points $g = \frac{1}{p}$.

2.2. Bulk and Boundary Operators

The loops in 2D quantum gravity may be considered as generating functions for the local operators of the theory[19]. Since the loops of the dense phase are clearly different from the ones in the standard lattice models it will be interesting to investigate their spectrum. Recall that Lian and Zuckerman have calculated the bulk spectrum in a version of continuum Liouville gravity. They found for the (p, q) minimal model coupled to gravity an infinite number of states with Liouville charges α given by (as always $p > q$)

$$\frac{\alpha}{\gamma} = \frac{p+q-k}{2q} \qquad k = 1, 2, \ldots \tag{2.12}$$

with the important restrictions $k \not\equiv 0(\text{mod q})$ and $k \not\equiv 0(\text{mod p})$ (γ is the charge of the identity operator, $k = p-q$). In [20],[21]it was shown that it is precisely these states which propagate around the torus. Now, in [8]it was found that the torus partition function for the dense minimal models coincides with the one found in [21], establishing that the dense models do have the expected bulk spectrum. As is well known, the standard matrix model realizations of the minimal models contains further operators. One way to see them is to look at the scaling dimensions obtained from expanding the macroscopic loops in the length ℓ. It is found that there are also operators with charges (2.12)but k divisible by p. They were identified by Martinec, Moore and Seiberg [22] as boundary operators. The first such operator, $k = p$, has $\alpha = \frac{\gamma}{2}$ and is beautifully interpreted as the length (ϕ is the Liouville field)

$$\ell = \oint e^{\frac{\gamma}{2}\phi} \tag{2.13}$$

For the dense phase models, the spectrum of dimensions found in the loop expansions was worked out in [16]. It is easily seen that they correspond to operators with $\frac{\alpha}{\gamma}$ as in (2.12)but with k divisible by p *missing* and k divisible by q *appearing*. The dense models are therefore different from the standard models because they possess a different set of boundary operators. This has dramatic effects, like the non-trivial Hausdorff dimension of the boundary. Indeed, here the first boundary operator has $\alpha = \frac{p}{q}\frac{\gamma}{2}$ and may be interpreted as the quantum length

$$\ell = \oint e^{\frac{p}{q}\frac{\gamma}{2}\phi} \tag{2.14}$$

The Hausdorff dimension $2g = 2\frac{q}{p}$ may be directly read off from the last equation. Note that the choice of α in (2.14)corresponds to taking the "wrong" branch in the KPZ dressing of the identity on the boundary. It appears that the Seiberg rule for choosing these branches does not necessarily hold for boundary operators [23][22]. It should also be pointed out that the dense minimal models possess $p-1$ loops as opposed to the dilute minimal models, which have $q-1$ loops. In the KP description of the minimal models inspired by the matrix chains the dense phase is formally recovered by interchanging the operators P and Q (see [4], [5],[24]). This remains however a formal excercise since it is not hard to see[3] that for the matrix chains the order of P is always larger than the order of Q (except for the hermitian one matrix model which is in the dense phase.). Our interpretation of dense models as being endowed with "standard worldsheets" but "nonstandard boundaries" is especially manifest in the example of the subsequent section.

2.3. $g = \frac{2}{3}$

$g = \frac{2}{3}$ is "pure gravity" ($C = 0$) in the dense phase. And indeed, for $n = 1$ in the maximally dense case (only $g_0 = 1 \neq 0$) we can integrate out the M matrix in (1.2)and obtain

$$Z = \int \mathcal{D}\Phi_1 \exp\{-N\mathrm{Tr}[\frac{1}{2}\Phi_1^2 - \frac{1}{4P_0^2}\Phi_1^4]\} \tag{2.15}$$

Of course the integration is this trivial only for the partition function and gets harder (although it may still be done) if the boundary is present: Remember the loop operator is $\frac{1}{N}\mathrm{Tr}e^{LM}$, *not* $\frac{1}{N}\mathrm{Tr}e^{L\Phi_1}$. It shows, however, that the *bulk* consists of ordinary φ^4 lattice gravity while the *boundary* is different.

The exact solution (2.11)to the loop equation (2.3)reads

$$\bar{W}^3 - (\frac{1}{3}(P_0^2 - 6) + \bar{P}^2)\bar{W} - \frac{2}{27}(P_0^2 - 6)^{\frac{3}{2}} + \frac{2}{3}P_0\bar{P}^2 = 0 \tag{2.16}$$

Being of third degree, it may be solved for $\bar{W}(\bar{P})$ to yield ($\Delta = P_0^2 - 6$)

$$\begin{aligned} &\bar{W}(\bar{P}) = \\ &- \left[\frac{1}{3}P_0\bar{P}^2 - \frac{1}{27}\Delta^{\frac{3}{2}} + \frac{1}{\sqrt{27}}\bar{P}\sqrt{2(P_0^2+3)\bar{P}^2 - \bar{P}^4 - \frac{1}{3}\Delta^2 - \frac{2}{3}P_0\Delta^{\frac{3}{2}}}\ \right]^{\frac{1}{3}} \\ &- \left[\frac{1}{3}P_0\bar{P}^2 - \frac{1}{27}\Delta^{\frac{3}{2}} - \frac{1}{\sqrt{27}}\bar{P}\sqrt{2(P_0^2+3)\bar{P}^2 - \bar{P}^4 - \frac{1}{3}\Delta^2 - \frac{2}{3}P_0\Delta^{\frac{3}{2}}}\ \right]^{\frac{1}{3}} \end{aligned} \tag{2.17}$$

---

[3] We acknowledge a discussion with M.R.Douglas on this matter.

2.4. $g=\frac{1}{3}$

This model is the first nontrivial point in the "topological" series $g=\frac{1}{p}$ and has $n=-1$. Because of the minus signs one easily proves that all *closed* string diagrams are identically zero because of precise cancellations. The free energy is zero even before taking the scaling limit! Open string diagrams, however, survive (see fig. 3). Just as $g=\frac{1}{2}$, the model is rather a theory of loops than of surfaces. The solution (2.11)of (2.3)is

$$\bar{W}^3-(P_0^2-2+\frac{1}{3}\bar{P}^2)\bar{W}+\frac{2}{3}(1+P_0^2)\bar{P}-\frac{2}{27}\bar{P}^3=0 \tag{2.18}$$

The equation may be solved to give ($\Delta=P_0^2-2$)

$$\begin{aligned}&\bar{W}(\bar{P})=\\&+\left[\frac{1}{27}\bar{P}^3-\frac{1}{3}(P_0^2+1)\bar{P}+\sqrt{\frac{1}{9}(P_0^2+1)^2\bar{P}^2-\frac{1}{27}\Delta^2\bar{P}^2-\frac{1}{27}P_0^2\bar{P}^4-\frac{1}{27}\Delta^3}\right]^{\frac{1}{3}}\\&+\left[\frac{1}{27}\bar{P}^3-\frac{1}{3}(P_0^2+1)\bar{P}-\sqrt{\frac{1}{9}(P_0^2+1)^2\bar{P}^2-\frac{1}{27}\Delta^2\bar{P}^2-\frac{1}{27}P_0^2\bar{P}^4-\frac{1}{27}\Delta^3}\right]^{\frac{1}{3}}\end{aligned} \tag{2.19}$$

It is manifest that the solution is analytic at $P_0^2=P_*^2=2$.

2.5. $g=\frac{1}{4}$

In order to study the puzzling question of the singularity in the lattice cosmological constant we have analyzed yet annother topological point. The model is less trivial than $g=\frac{1}{3}$ because here $n=-\sqrt{2}$ and we do not have precise cancellation of the worldsheet-loops away from the scaling region. The exact solution for $\bar{W}(\bar{P})$ is (we introduced the abbreviations $\sigma_\pm=1\pm\frac{1}{2}\sqrt{2}$)

$$\bar{W}(\bar{P})=-\sqrt{\sigma_+(P_0^2-4\sigma_-)+\sigma_-\bar{P}^2-\frac{1}{2}\sqrt{2}\sigma_-(\bar{P}-B)\sqrt{(\bar{P}-\bar{P}_R)(\bar{P}-\bar{P}_L)}} \tag{2.20}$$

which is, in accordance with (2.11)a solution to a bicubic equation for $\bar{W}$. The positions of the cut are given through $\bar{P}_R+\bar{P}_L=4\sqrt{2}\sigma_+P_0-2B$ and $\bar{P}_R\bar{P}_L=4\sigma_+^4(P_0^2-4\sigma_-)^2\frac{1}{B^2}$. The auxiliary parameter B is determined by

$$B^4-\frac{8}{3}\sqrt{2}\sigma_+P_0B^3+\frac{4}{3}\sigma_+^2(P_0^2-4\sigma_-)B^2-\frac{4}{3}\sigma_+^4(P_0^2-4\sigma_-)^2=0 \tag{2.21}$$

It is seen[4] that any singular behavior in the cosmological constant P_0 would have to be generated from eq.(2.21). A (most conveniently numerical) study shows however that the branchpoints of the *physical* sheet of (2.21)are *not* located at the critical value $P_*^2 = 4\sigma_-$ (i.e.$P_0 \approx 1.082$) but are lying off the real axis at $\pm 0.327 \pm 0.633i$. This confirms the general discussion given in section 2.1. and agrees with the picture of [18]. Let us further note that, since there are no singularities on the real P_0 axis, the lattice model does not undergo *any* phase transition in the bulk cosmological constant, i.e. not even for some $P_0 < P_*$.

3. The Multicritical Phases

3.1. Solution and Scaling of the Loop Equation

We will now turn on $m-1$ couplings $\{g_k\}$ in order to reach the m^{th} multicritical phase. The loop equation is easily generalized to

$$\begin{aligned} &W(P)^2 + W(2P_0 - P)^2 + nW(P)W(2P_0 - P) = \\ &= V'(P)W(P) + V'(2P_0 - P)W(2P_0 - P) + \sum_{k=1}^{m}\sum_{j=1}^{k} g_{k+1}W_{j-1}[P^{k-j} + (2P_0 - P)^{k-j}] \end{aligned} \tag{3.1}$$

where V' is the derivative of the potential (1.6). A new feature compared to the dense phase equation (2.2)is the appearance of the (as yet) unknown functions $\{W_j\}$ (see eq.(1.8)) which depend on all matter couplings $\{g_k\}$,P_0. As for the dense phase the linear terms may be eliminated through the transformation $P = P_0 + \bar{P}$, $W(P) = \frac{1}{2\sin^2\pi g}[V'(P) + \cos\pi g\ V'(2P_0 - P)] + \bar{W}(\bar{P})$:

$$\bar{W}(\bar{P})^2 + \bar{W}(-\bar{P})^2 + n\bar{W}(\bar{P})\bar{W}(-\bar{P}) = \sum_{i=0}^{m} s_{2i}\bar{P}^{2i} \tag{3.2}$$

Here the $\{s_{2i}\}$ are both explicitly and implicitly (through the functions $\{W_j\}$) dependent on the matter couplings. In the m^{th} critical phase the loop function should behave for $\bar{P} \to 0$ as $W(\bar{P}) \sim \bar{P}^g$ with $m-1 \le g \le m$. The critical point is found to be fixed through the conditions[5]

$$s_0 = s_2 = \ldots = s_{2(m-1)} = 0 \qquad s_{2m} \neq 0 \tag{3.3}$$

[4] That is because the j-leg functions $\{W_j\}$ are simple rational functions in B, as (2.20)shows. E.g., W_1 is given by $P_0 + \frac{1}{B^2}[\frac{1}{4}\sqrt{2}B^5 - (1+\frac{1}{8}\sqrt{2})P_0B^4 - 2\sigma_+P_0^2B^3 + (2\sigma_+^4(P_0^2 - 4\sigma_-)P_0 - (\frac{25}{2} + \frac{35}{4}\sqrt{2})P_0^3)B^2 - \sqrt{2}\sigma_+^4(P_0^2 - 4\sigma_-)^2B - (\frac{3}{4}\sqrt{2}+1)\sigma_+^2(P_0^2 - 4\sigma_-)^2P_0]$

[5] Note that we have m couplings at our disposal in order to fulfill the m conditions.

At that point eq.(3.2)may be explicitly solved; the solution with the correct asymptotics at $\bar{P} = \infty$ reads

$$\begin{aligned} \bar{W}(\bar{P}) = -\frac{1}{2\sin\pi g}\frac{1}{\sqrt{2-n}} g^*_{m-1}\bar{P}^m & \Bigg[\left(-\frac{\bar{P}_L}{\bar{P}} + \sqrt{(\frac{\bar{P}_L}{\bar{P}})^2 - 1} \right)^{m-g} \\ & + \left(-\frac{\bar{P}_L}{\bar{P}} - \sqrt{(\frac{\bar{P}_L}{\bar{P}})^2 - 1} \right)^{m-g} \Bigg] \end{aligned} \tag{3.4}$$

The position of the left branchpoint $\bar{P}_L$ as well as the critical couplings $\{g^*_k\}$ are determined from the asymptotics. We see that indeed $W(\bar{P}) \sim \bar{P}^g$. In view of this solution at the m-th multicritical point our arguments about the structure of the Riemann surface of $\bar{W}$ in section 2.1. may be repeated; we thus conclude that the exact solution for *rational* $g = \frac{p}{q}$ and arbitrary matter couplings is given by

$$\bar{W}^q + h_{q-1}(\bar{P})W^{q-1} + \ldots + h_1(\bar{P})\bar{W} + h_0(\bar{P}) = 0 \tag{3.5}$$

Again the $h_i(\bar{P})$ are polynomials in $\bar{P}$ whose degree increases with p; they (as well as the functions W_j) are determined by matching (3.5)and (3.2)at $\bar{P} = \infty$. In order to infer the scaling limit of $\bar{W}(\bar{P})$ we however do not need this exact solution; instead we simply generalize the arguments of [15], [16]: upon scaling $\bar{P} = \bar{a}z$ one concludes from (3.4)that $\bar{W}(\bar{P}) \sim \bar{a}^g w(z)$; thus the dominant piece in (3.2)is s_0 which has to vanish as $\bar{a}^{2g}$. Parametrizing the proper approach to the critical point (3.3)by the continuum cosmological constant Λ one expects to find $s_0 \sim \bar{a}^{2g}\Lambda^g$; this will give as expected $w(z) = \frac{-1}{\sin\pi g}[(z+\sqrt{z^2-\Lambda})^g + (z-\sqrt{z^2-\Lambda})^g]$. The Hausdorff dimension takes on the "classical" value 2 for all multicritical phases.

3.2. Boltzmann Weights on Flat and Random lattices

The dense and dilute loop gas has been constructed as a direct adaptation of flat lattice models to the random case. It is therefore natural to ask whether we can go back from our formulation of the multicritical loop gas and deduce how a multicritical $O(n)$ model might look like on *flat* honeycomb lattices. Generically we would expect to obtain such a model by introducing interactions between the loops [14], [13]. In the case of the first multicritical phase ($m = 3$) a concrete suggestion may be made, using the procedure of [2]. In this case the matrix model (1.2)generating the loop gas may be rewritten with the help of an additional matrix Ψ as

$$
\begin{aligned}
Z = & \int \mathcal{D}M\mathcal{D}\Psi \prod_{i=1}^{n} \mathcal{D}\Phi_i \times \\
& \times \exp\{-N\mathrm{Tr}[\frac{1}{2}\mathrm{M}^2 + \frac{\mathrm{g}_1}{3}\mathrm{M}^3 + \frac{1}{2}\Psi^2 + \mathrm{i}\sqrt{\frac{\mathrm{g}_2}{2}}\Psi\mathrm{M}^2 + \frac{1}{2}\sum_{\mu=1}^{\mathrm{n}}\Phi_\mu^2 - \frac{1}{2\mathrm{P}_0}\sum_{\mu=1}^{\mathrm{n}}\Phi_\mu^2\mathrm{M}]\}
\end{aligned}
\tag{3.6}
$$

which is seen to possess the graphical expansion exemplified in fig. 4. In addition to the loops we have dimers placed on the random “honeycomb” lattice. Their fugacity has to be chosen negative in order to perturb away from the $m = 2$ regime. Note that this formulation may be applied to flat lattices and, even though it does not constitute an exactly solvable model, could be checked by numerical analysis of the transfer matrix. According to our phase diagram in fig. 2 we predict central charges between $C = -2$ and $C = -7$. For the higher phases it is less clear how to proceed. One guess would be to complement the dimers by increasingly longer strands (2-chains, 3-chains etc.) with alternating signs for their Boltzmann weights. An interesting question from the point of view of polymer theory is whether the multicritical matter induces in the continuum limit an effective interaction between the loops. An alternative possibility is that directly introducing such interactions results in yet another class of multicritical polymers.

3.3. Geometrical critical exponents for the $O(n)$ vector model

The order parameters (the magnetic operators) in the $O(n)$ model have a simple discription in terms of the loop gas. The m-th magnetic operator Ψ_L is represented as the source of L nonintersecting lines meeting at a point. The correlation function of two such operators can be evaluated as the partition function F_L of a network consisting of L nonintersecting lines tied at their extremities, moving in the sea of vacuum loops of the $O(n)$ model [25]. The dimension of the star operator Ψ_L can be extracted from the dimension of the partition function F_L. This last quantity can be calculated immediately.

First observe that the L nonintersecting lines cut the world sheet into k pieces with the topology of a disk. Let $\ell_1, ..., \ell_L$ be the lengths of the L lines which form the watermelon network. As usual, we first sum over all configurations of the world sheet populated by vacuum loops keeping these lengths fixed. Then the partition function F_L can be represented as an integral over $\ell_1, ...\ell_L$ of the product of L loop amplitudes with lengths $\ell_1 + \ell_2, \ell_2 + \ell_3, ..., \ell_L + \ell_1$

$$F_L = \int \prod_{i=1}^{k} d\ell_i e^{-2P_0\ell_i} \quad \tilde{w}(\ell_1+\ell_2)\tilde{w}(\ell_2+\ell_3)...\tilde{w}(\ell_L+\ell_1)$$

By the general scaling argument

$$F_L \sim \Lambda^{2\delta_L - \gamma_{str}}$$

where δ_L is the gravitational dimension of the operator Ψ_L. On the other hand the loop amplitude behaves asymptotically as $w(\ell) \sim \ell^{-1-g}$ and therefore

$$F_L \sim M^{Lg} = \Lambda^{Lg/(2\nu)}$$

It follows that

$$\delta_L = (Lg/(2\nu) + \gamma_{str})/2 = \begin{cases} \frac{Lg}{4} - \frac{g-1}{2}, & \text{if } g > 1 \\ \frac{L}{4} - \frac{1-g}{2g}, & \text{if } g < 1 \end{cases}$$

The corresponding flat conformal dimensions are

$$\Delta_L = \begin{cases} \Delta_{L/2,0} = \frac{g\ L^2}{16} - \frac{(g-1)^2}{4g}, & \text{if } g > 1 \\ \Delta_{0,L/2} = \frac{L^2}{16g} - \frac{(g-1)^2}{4g}, & \text{if } g < 1 \end{cases}$$

The smallest dimension Δ_1 is positive only in the interval $-1/2 \le g \le 2$. Outside this interval the propagator of the nonintersecting random walk grows with the distance between its two extremities. The fact that two points are connected with a line leads to an effective repulsion between them. Such a phenomenon is typical for nonunitary theories.

4. The Phase Boundaries

4.1. Exact Solution and Scaling of the Loop Equation at $n = \pm 2$

We will now investigate in some detail the loop gas with $n = \pm 2$ in the presence of multicritical matter. This will serve as an important check of our ideas. As we discussed before we expect to be able to tune to points with $g = p$ integer, which form the crossover points from the p-phase to the $(p+1)$-phase. Consider again the scaled loop function

$$w(z) = \frac{-1}{\sin \pi g}[(z + \sqrt{z^2 - M^2})^g + (z - \sqrt{z^2 - M^2})^g] \tag{4.1}$$

Were it not for the "wavefunction renormalization" factor $\frac{1}{\sin \pi g}$ we would get a trivial (i.e. purely analytic) result for $w(z)$ for $g \in N$. Including this factor and subtracting terms analytic in z we instead obtain

$$w(z) = \frac{(-1)^{g-1}}{\pi}[(z + \sqrt{z^2 - M^2})^g - (z - \sqrt{z^2 - M^2})^g] \log \frac{z + \sqrt{z^2 - M^2}}{M} \tag{4.2}$$

Upon Laplace-transforming this result, one obtains correctly $w(\ell) = \frac{1}{\ell} M^g K_g(M\ell)$, as one expects from Liouville theory [11]. We will now demonstrate how to derive this result from our lattice model. The algebraic approach clearly breaks down for integer g: the Riemann surface associated with $W(P)$ becomes ininitely foliated. Fortunately it is however possible to directly solve the saddlepoint equation of the $O(n)$ model's matrix model formulation [26]. This equation for the eigenvalue density of the matrix model reads [17]

$$\int_a^b dy \rho(y)(\frac{1}{x-y} + \frac{n}{2}\frac{1}{2P_0 - x - y}) = \frac{1}{2}V'(x) = \frac{1}{2}\sum_{k=1}^{m} g_{k+1} x^k \tag{4.3}$$

where $V(x)$ is the potential of the matrix model. It is convenient to change variables to $\lambda' = \frac{1}{2}(P_0 - x)$, $\mu' = \frac{1}{2}(P_0 - y)$, $b_0 = \frac{1}{2}P_0$, $\rho(x) \to \frac{1}{2}\rho(\lambda')$, $g_k \to (-\frac{1}{2})^{k-2} g_k$ and rewrite (4.3)as

$$\begin{aligned} \int_{a'}^{b'} \rho(\mu') \frac{2\mu'}{\lambda'^2 - \mu'^2} &= 2\sum_{k=1}^{m} g_{k+1}(\lambda' - b_0)^k \qquad (n = +2) \\ \int_{a'}^{b'} \rho(\mu') \frac{2\lambda'}{\lambda'^2 - \mu'^2} &= 2\sum_{k=1}^{m} g_{k+1}(\lambda' - b_0)^k \qquad (n = -2) \end{aligned} \tag{4.4}$$

A further transformation $\lambda' = \sqrt{A + B\lambda}$ with $A = \frac{1}{2}(a'^2 + b'^2)$ and $B = \frac{1}{2}(a'^2 - b'^2)$ simplifies these equations to

$$\begin{aligned} \int_{-1}^{1} d\mu \frac{\rho(\mu)}{\lambda - \mu} &= 2\sum_{k=1}^{m} g_{k+1}(\sqrt{A + B\lambda} - b_0)^k \qquad (n = +2) \\ \int_{-1}^{1} d\mu \frac{\rho(\mu)}{\sqrt{A + B\mu}} \frac{1}{\lambda - \mu} &= 2\sum_{k=1}^{m} \frac{g_{k+1}}{\sqrt{A + B\lambda}}(\sqrt{A + B\lambda} - b_0)^k \qquad (n = -2) \end{aligned} \tag{4.5}$$

These integral equations may now be immediately inverted [27] to yield

$$\begin{aligned} \rho(\lambda) &= -\frac{2}{\pi^2}\sqrt{1 - \lambda^2} \int_{-1}^{1} d\mu \frac{1}{\lambda - \mu} \frac{1}{\sqrt{1 - \mu^2}} \sum_{k=1}^{m} g_{k+1}(\sqrt{A + B\mu} - b_0)^k \\ \rho(\lambda) &= -\frac{2}{\pi^2}\sqrt{A + B\lambda}\sqrt{1 - \lambda^2} \int_{-1}^{1} d\mu \frac{1}{\lambda - \mu} \frac{1}{\sqrt{1 - \mu^2}} \sum_{k=1}^{m} \frac{g_{k+1}}{\sqrt{A + B\lambda}}(\sqrt{A + B\mu} - b_0)^k \end{aligned} \tag{4.6}$$

for $n=+2$ and $n=-2$, respectively. Since $\rho(\lambda)$ has to be non-negative it is easy to prove the positivity condition

$$\begin{aligned}
&\int_{-1}^{1} d\mu \frac{1}{\sqrt{1-\mu^2}} \sum_{k=1}^{m} g_{k+1}(\sqrt{A+B\mu}-b_0)^k = 0 \qquad (n=+2) \\
&\int_{-1}^{1} d\mu \frac{1}{\sqrt{1-\mu^2}} \frac{1}{\sqrt{A+B\mu}} \sum_{k=1}^{m} g_{k+1}(\sqrt{A+B\mu}-b_0)^k = 0 \qquad (n=-2)
\end{aligned} \tag{4.7}$$

Together with the normalization condition for the density $\int_{a'}^{b'} d\mu' \rho(\mu') = 1$, i.e.

$$\frac{B}{2} \int_{-1}^{1} d\mu \frac{\rho(\mu)}{\sqrt{A+B\mu}} = 1 \qquad (n=\pm 2) \tag{4.8}$$

we are thus given two constraints determining the eigenvalue interval $[a', b']$ as a function of the couplings $\{b_0, g_k\}$. Equations (4.6),(4.7),(4.8) constitute the exact solution of the $n=\pm 2$ loop equation in the presence of general multicritical matter. We will now investigate the singular structure of the solution; specific examples are presented in the next section. The critical points are located by setting $b'=0$ [17]. At the g^{th} multicritical point we expect the density to behave for $\lambda' \to 0$ like

$$\rho(\lambda') \sim \lambda'^g \tag{4.9}$$

In order to locate these points we rewrite (4.6)for $n=+2$ as

$$\begin{aligned}
\rho(\lambda) &= \frac{2}{\pi}\sqrt{1-\lambda^2} \sum_{i=0}^{[\frac{m-2}{2}]} r_{2i}(A+B\lambda)^i \\
&- \frac{2}{\pi^2}\sqrt{1-\lambda^2} \sum_{l=0}^{[\frac{m-1}{2}]} t_{2l+1}(A+B\lambda)^l \int_{-1}^{1} d\mu \frac{1}{\lambda-\mu} \frac{\sqrt{A+B\mu}}{\sqrt{1-\mu^2}}
\end{aligned} \tag{4.10}$$

and for $n=-2$ as

$$\begin{aligned}
\rho(\lambda) &= \frac{2}{\pi}\sqrt{1-\lambda^2} \sum_{i=0}^{[\frac{m-3}{2}]} r_{2i+1}(A+B\lambda)^{i+\frac{1}{2}} \\
&- \frac{2}{\pi^2}\sqrt{1-\lambda^2} t_0 \sqrt{A+B\lambda} \int_{-1}^{1} d\mu \frac{1}{\lambda-\mu} \frac{1}{\sqrt{1-\mu^2}} \frac{1}{\sqrt{A+B\mu}} \\
&- \frac{2}{\pi^2}\sqrt{1-\lambda^2} \sum_{l=0}^{[\frac{m-2}{2}]} t_{2l+2}(A+B\lambda)^{l+\frac{1}{2}} \int_{-1}^{1} d\mu \frac{1}{\lambda-\mu} \frac{\sqrt{A+B\mu}}{\sqrt{1-\mu^2}}
\end{aligned} \tag{4.11}$$

Here the $\{r_{2i}, t_{2l+1}\}$ and $\{r_{2i+1}, t_{2l}\}$ are known functions of the couplings $\{b_0, g_k\}$ and the parameters A, B (but not of λ !) whose precise form we do not need at present. The remaining integrals in (4.10),(4.11)may be expressed in terms of complete elliptic integrals of the third kind. At the critical point $b' = 0$ they become elementary. Transforming back to λ' (4.10) may then be written

$$\rho(\lambda') = \frac{4}{\pi a'^2}\sqrt{a'^2 - \lambda'^2}\sum_{i=0}^{[\frac{m-2}{2}]} r_{2i}\lambda'^{2i+1} - \frac{4}{\pi^2}\sum_{l=0}^{[\frac{m-1}{2}]} t_{2l+1}\lambda'^{2l+1}\log\frac{\lambda'}{a' + \sqrt{a'^2 - \lambda'^2}} \tag{4.12}$$

In view of (4.9), the possible critical behavior for $n = +2$ is now transparent from (4.12). We note that it is immediately obvious that we can reach only points with g odd, as expected. There are two kinds of critical points:

1. $m = g$ (we approach the g^{th} point from *below.*) It is obtained through the conditions

$$\begin{aligned} &r_0 = r_2 = \ldots = r_{g-3} = 0 \\ &t_1 = t_3 = \ldots = t_{g-2} = 0 \qquad t_g \neq 0 \end{aligned} \tag{4.13}$$

There are $g - 1$ couplings $\{g_k\}$ and $g - 1$ constraints, so we indeed have a critical *point.*

2. $m = g + 1$ (we approach the g^{th} point from *above.*) The conditions are

$$\begin{aligned} &r_0 = r_2 = \ldots = r_{g-3} = 0 \qquad r_{g-1} \neq 0 \\ &t_1 = t_3 = \ldots = t_{g-2} = t_g = 0 \end{aligned} \tag{4.14}$$

Here one has g couplings $\{g_k\}$ and g constraints; (4.14)defines again a critical point. However, one may imagine relaxing the last condition to $t_g \neq 0$ and still be consistent with the scaling law (4.9), see below.

An analogous analysis shows that for $n = -2$ the points with even g are generated. At $b' = 0$ (4.11) gives

$$\rho(\lambda') = \frac{4}{\pi a'^2}\sqrt{a'^2 - \lambda'^2}\sum_{i=0}^{[\frac{m-3}{2}]} r_{2i+1}\lambda'^{2i+2} - \frac{4}{\pi^2}\sum_{l=0}^{[\frac{m-2}{2}]} t_{2l+2}\lambda'^{2l+2}\log\frac{\lambda'}{a' + \sqrt{a'^2 - \lambda'^2}} \tag{4.15}$$

The first integral in (4.11)is actually divergent for $b' \to 0$. This simply gives the condition $t_0 = 0$. Again one finds two distinct kinds of critical points:

1. $m = g$ (we approach the g^{th} point from *below.*) It is obtained through the conditions

$$\begin{aligned} &r_1 = r_3 = \ldots = r_{g-3} = 0 \\ &t_0 = t_2 = \ldots = t_{g-2} = 0 \qquad t_g \neq 0 \end{aligned} \tag{4.16}$$

2. $m = g+1$ (we approach the g^{th} point from *above*.) The conditions are

$$\begin{aligned} &r_1 = r_3 = \ldots = r_{g-3} = 0 \qquad r_{g-1} \neq 0 \\ &t_0 = t_2 = \ldots = t_{g-2} = t_g = 0 \end{aligned} \tag{4.17}$$

Here also one has as many constraints as couplings, but it is consistent to relax $t_g = 0$ in 2..

Having found the critical points we will now scale the density in the vicinity of these points: Introduce a cutoff $\bar{a}$ and define $\frac{\lambda'}{a'} = \bar{a}\zeta$ in addition to the usual scaled separation of the cuts $k' = \frac{b'}{a'} = \bar{a}M$. First concentrate on the points of type 2. Approaching them along a line in coupling constant space where the conditions (4.14), (4.17)remain satisfied we obtain from (4.10), (4.11)for the singular limit of $\rho(\lambda') \sim \bar{a}^g \rho(\zeta)$

$$\rho(\zeta) = \zeta^{g-1}\sqrt{\zeta^2 - M^2} \tag{4.18}$$

$\bar{a}^g w(z)$ being defined as the singular part of $W(P) = W(P') = -\frac{1}{2}\int_{b'}^{a'} d\mu' \frac{\rho(\mu')}{P'-\mu'}$ (here $P' = \frac{1}{2}(P_0 - P)$, $\frac{P'}{a'} = -\bar{a}z$) we derive from (4.18)

$$w(z) = (-1)^{g-1} z^{g-1}\sqrt{z^2 - M^2}\log\frac{z + \sqrt{z^2 - M^2}}{M} \tag{4.19}$$

Some divergent contributions analytic in z, M were discarded in the course of the derivation; the general meaning of such terms was elucidated in [11]. Eq.(4.19)is already almost (4.2). For $g = 1, 2$ they in fact exactly coincide. (Overall wavefunction normalizations are ignored.) For $g \geq 3$ there are additional terms of the form $z^i M^{g-1-i}\sqrt{z^2 - M^2}\log\ldots$ in (4.2). In view of eqs.(4.10), (4.11)it is clear how to generate these terms. Approaching the critical point we have to *tune* the $\{g_k\}$ such that the $\{r_{2i}\}$, $\{r_{2i+1}\}$ vanish as the appropriate power of k'. This constitutes, for g integer, the analog of the "analytic redefinitions" of [11]in the half-integer case. Only along special trajectories in coupling constant space do we satisfy the Wheeler-de-Witt constraint.

It remains to analyze the relation between the cosmological constant Λ and the scaling parameter M. This is done by investigating the conditions (4.7), (4.8). The result is[6]

$$\begin{aligned} &\Lambda = M^2 \log\frac{1}{\bar{a}M} \qquad (g = 1) \\ &\Lambda = M^2 \qquad (g \geq 2) \end{aligned} \tag{4.20}$$

[6] We have not carried out a detailed proof except for $g = 1, 2, 3, 4$ (see examples) but strongly suspect the validity of our claim for all g.

The critical points of type 1 are essentially different from the type 2 just discussed. They exhibit logarithmic behavior in the eigenvalue density; it is easily seen that upon scaling we *do not* obtain (4.2)and (4.20)does not hold.

The general theory of this section will now be applied to the lowest (and most interesting) values of g.

4.2. $g = 1$

This is the loop gas model at $C = 1$. $m = 1$ is the right boundary of the dense phase discussed in section 2; $t_1 = 1$. The critical coupling is $b_* = 2$. If we introduce the cosmological constant as $b_0^2 = 2 + 32\bar{a}^2\Lambda$ the relation to the scaling parameter is $\Lambda = (M \log \bar{a}M)^2$. The density, however, contains a logarithmic piece, as explained above.

On the other hand, $m = 2$ constitutes the left boundary of the dilute phase. The parameters in this case read $t_1 = 1 - 2g_1 b_0$ and $r_0 = g_1 B$. Imposing the condition (4.14), i.e. $g_1 = \frac{1}{2b_0}$, the logarithmic piece in the density is killed and one obtains

$$\rho(\lambda) = \frac{1}{\pi b_0}\sqrt{(a'^2 - \lambda'^2)(\lambda'^2 - b'^2)} \tag{4.21}$$

Working out the positivity condition (4.7)gives $A = b_0^2$, while the normalization constraint yields

$$\frac{2}{3\pi b_0}[Aa'E(k') - (A^2 - B^2)\frac{1}{a'}K(k')] = 1 \tag{4.22}$$

$K(k')$, $E(k')$ are the standard complete elliptic integrals of the first and second kind, respectively. Considering the limit $k' \to 0$ of (4.22)one locates the critical coupling to be $b_0^2 = \frac{3\pi}{2\sqrt{2}}$; more importantly, setting $b_0^2 = \frac{3\pi}{2\sqrt{2}}(1 + \bar{a}^2\Lambda)$ one finds $\bar{a}^2\Lambda = \frac{3}{2}k'^2 \log\frac{4}{k'}$, hence proving the first assertion in (4.20). Note that the Boltzmann weights of the model are all *positive*, as is expected for a unitary theory[7].

4.3. $g = 2$

Here the central charge is $C = -2$. $m = 2$ is the right boundary of the dilute phase. The parameters are [8] $t_0 = -b_0 + g_1 b_0^2$ and $t_2 = g_1$. Imposing (4.16)gives $g_1 = \frac{1}{b_0}$. The

[7] Be aware of our redefinition of the couplings $\{g_k\}$ following eq. (4.3).

[8] Note that for $g_1 = 0$ we are in the dense phase ($m = 1$). The "critical limit" thus corresponds to $b_0 \to 0$, meaning infinite Boltzmann weights for loops. In other words, it does not correspond to a critical theory at all; the lattice version of "$C = -\infty$".

density contains a logarithmic term. The positivity condition gives $2a'E(k') = \pi b_0$ while the normalization condition turns out to be identical to (4.22). One thus locates the critical point to be at $b_0^2 = \frac{24}{\pi^2}$ and finds $\bar{a}^2\Lambda = \frac{5}{2}k'^2 \log\frac{4}{k'}$. Therefore the Boltzmann weights are positive (a necessary consistency check since we are approaching this point from the dilute phase); however, there exists a logarithmic scaling violation $\Lambda = M^2 \log\frac{1}{\bar{a}M}$.

$m = 3$ corresponds to the left boundary of the first higher multicritical phase (the Yang-Lee phase). One has $t_0 = -b_0 + g_1 b_0^2 - g_2 b_0^3$, $t_2 = g_1 - 3g_2 b_0$ and $r_1 = Bg_2$. Imposing $t_0 = t_2 = 0$ it follows

$$\rho(\lambda) = \frac{1}{\pi b_0^2}\lambda'\sqrt{(a'^2-\lambda'^2)(\lambda'^2-b'^2)} \tag{4.23}$$

Positivity gives $A = b_0^2$ and normalization $B = 2b_0$. Thus

$$k' = \sqrt{\frac{b_0-2}{b_0+2}} \tag{4.24}$$

and it is obvious ($b_* = 2$) that $\Lambda = M^2$. The couplings are $g_1 = \frac{3}{2b_0}$, $g_2 = \frac{1}{2b_0^2}$. This translates into positive weights for the φ^3 vertices and negative weights for the φ^4 vertices, as it must be in the Yang-Lee phase. Surprisingly, the model is *exactly* identical to the "$D = -2$" theory solved several years ago [28]. It possesses a Parisi-Sourlas supersymmetry and is most elegantly described by a zero dimensional supersymmetric matrix model. The supermatrix is $\Sigma = M + \bar{\theta}\Psi + \Psi^+\theta + \bar{\theta}\theta A$ where θ,$\bar{\theta}$ are Grassmann variables, Ψ,Ψ^+ are Grassmann valued hermitian matrices and A is the auxiliary hermitian matrix. The action is[9]

$$S = N\mathrm{Tr}\int d\theta d\bar{\theta}[\frac{1}{2}(\frac{\partial}{\partial\theta}\Sigma)(\frac{\partial}{\partial\bar{\theta}}\Sigma) + \frac{1}{2}\Sigma^2 - \frac{g}{3}\Sigma^3] \tag{4.25}$$

Upon performing the Grassmann integrals and integrating out the auxiliary field one obtains

$$S = N\mathrm{Tr}[\frac{1}{2}(M - gM^2)^2 - \Psi^+\Psi(1-2gM)] \tag{4.26}$$

By looking at the diagrams the action S is generating one finds the graphical expansion

$$\sum_{\{\varphi^3\text{graphs}\}} g^v \sum_{\{\text{dimers } + \text{ loops}\}} 3^{v_0}(-1)^{\#(\text{dimers})}(-2)^{\#(\text{loops})} \tag{4.27}$$

[9] One may equally well use a Gaussian propagator.

where v is the total number of vertices and v_0 the number of vertices not occupied by either loops or dimers[10]. The dimers and loops are totally self-avoiding (see fig. 4). Now, a comparison of our loop gas Boltzmann weights with (4.27)immediately reveals the identity of the two models (the two lattice cosmological constants being related through $g = \frac{1}{4b_0}$). It is interesting to note that the loops are generated by the "fermionic fields" Ψ,$\bar{\Psi}$ while the dimers are associated with the auxiliary field A. What we have shown here is that the latter degrees of freedom are absolutely necessary in order to preserve the supersymmetry: without them, the critical behavior is different.

4.4. $g \geq 3$

We will only briefly comment on the type 2 critical point $m = 4$, a $g = 3$ model. The conditions that turn off the logarithmic terms are $t_1 = 1 - 2g_1 b_0 + 3g_2 b_0^2 - 4g_3 b_0^3 = 0$ and $t_3 = g_2 - 4g_3 b_0 = 0$. The third parameter is $r_0 = g_1 - 3g_2 b_0 + 6g_3 b_0^2 + g_3 A$ but now we have to remember that $r_0 = 0$ upon approaching criticality gives (4.19)while we produce (4.2)by *tending* r_0 to zero appropriately. It is possible if somewhat tedious to prove $\bar{a}^2\Lambda = \frac{5}{4}k'^2$ and therefore confirm (4.20).

We also carefully investigated $g = 4$; no new elements or surprises are found.

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[10] Because of the zero mode of the discrete Laplacian (4.27)is of course identically zero for closed string diagrams. Inserting a loop operator $\frac{1}{N}\mathrm{Tr}e^{LM}$ fixes the zero mode and yields a nontrivial result.

References

[1] V.Kazakov, Mod.Phys.Lett A4 (1989) 2125.

[2] M. Staudacher,Nucl. Phys. B 336 (1990) 349.

[3] E. Brézin, M. Douglas, V. Kazakov and S. Shenker, Phys. Lett. B237 (1990) 43; D. Gross and M. Migdal, Phys. Rev. Lett. 64 (1990) 717; Č. Crnković, P. Ginsparg and G. Moore, Phys. Lett. 237B (1990) 196.

[4] M. Douglas, Phys. Lett 238B (1990) 176.

[5] M.Douglas, "The two-matrix model", to appear in the proceedings of the 1990 Cargèse workshop; E.Martinec, Comm.Math.Phys. 138 (1991) 437; T.Tada, Phys.Lett. B259 (1991) 442.

[6] I. Kostov, Mod. Phys. Lett. A4 (1989) 217

[7] I. Kostov, Nucl. Phys. B 326 (1989)583.

[8] I.K Kostov, "Strings with Discrete Target Space," Saclay preprint SPhT/91-142.

[9] B. Nienhuis, in Phase Transitions and Critical Phenomena, Vol. 11, C. C. Domb and J. L. Lebowitz, eds., Ch. 1 (Academic Press, 1987)

[10] B. Eynard and J. Zinn-Justin, in preparation

[11] G.Moore, N.Seiberg and M. Staudacher, Nucl. Phys. B 362 (1991)665.

[12] J.Distler, Nucl. Phys. B342 (1990) 523.

[13] H.Saleur, "Polymers and Percolation in Two Dimensions and Twisted $N = 2$ Supersymmetry," Yale preprint YCTP-P38-91; H.Saleur, "Geometrical Lattice Models for $N = 2$ Supersymmetric Theories in Two Dimensions," Yale preprint YCTP-P39-91.

[14] A.B. Zamolodchikov, unpublished.

[15] I.Kostov, "Strings embedded in Dynkin diagrams", Lecture given at the Cargese meeting, Saclay preprint SPhT/90-133.

[16] I.K. Kostov, Phys. Lett. B 266 (1991) 42

[17] M.Gaudin and I.Kostov, Phys.Lett. B220(1989)200.

[18] H.Saleur, Mod.Phys.Lett. A6 (1991) 1709.

[19] T. Banks, M. Douglas, N. Seiberg and S. Shenker, Phys. Lett. 238B (1990) 279.

[20] D.Kutasov and N.Seiberg, Nucl.Phys. B358 (1991) 600

[21] M. Bershadski, I. Klebanov, Nucl. Phys. B 360 (1991) 559.

[22] E.Martinec, G.Moore and N.Seiberg, Phys.Lett. B263 (1991) 190.

[23] N. Seiberg,"Notes on Quantum Liouville Theory and Quantum Gravity," Rutgers preprint RU–90–29, to appear in the proceedings of the 1990 Yukawa International Seminar, and in the proceedings of the Cargèse Workshop on Random Surfaces.

[24] P. Di Francesco and D. Kutasov, Nucl. Phys. B342 (1990) 589; P. Di Francesco and D. Kutasov,"Integrable Models of Two–Dimensional Quantum Gravity,"Princeton preprint PUPT–1206 (1990), presented at the Cargèse Workshop.

[25] B. Duplantier and I. Kostov, Nucl Phys.B340 (1990) 491.
[26] M.Gaudin,unpublished.
[27] N.Muskhelishvili, Singular Integral Equations (Nordoff NV, Groningen, Netherlands, 1953).
[28] D. Boulatov, V. Kazakov, I. Kostov and A. Migdal, Nucl. Phys. B257 (1985) 433; F. David, Phys. Lett. 159B (1985) 303; I. Kostov and M. Mehta, Phys. Lett. 189B (1987) 118.

Figure Captions

Fig. 1. A configuration of nonintersecting loops

Fig. 2. The multicritical phase diagram of the loop gas coupled to 2D gravity. C is the central charge, g is the Coulomb gas coupling and n the weight of the loops.

Fig. 3. An example for a diagram of the topological model $(1, 3)$.

Fig. 4. A typical diagram in the $m = 3$ multicritical phase.